Algebraic Number Theory

Exercise Sheet 5

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Winter Semester 2024-25
19.11.2024

Exercise 1. Let $\mathbb{Q} \subset L$ be a number field and let α be an element in L, which is integral over $\mathbb{Z}$. Let $P(X)$ be the minimal polynomial of α over $\mathbb{Q}$. Recall that $P(X) \in \mathbb{Z}[X]$.
Show that $\mathbb{Z}[X]/(P(X)) \simeq \mathbb{Z}[\alpha] \subset L$.

Exercise 2. Let A be a ring and let I, J be two ideals in A.
(1) Let $f : A \to B$ be a surjective ring homomorphism. Show that $f(I)$ is an ideal in B.
(2) Let $\varphi : A \to A/I$ be the canonical ring homomorphism. Show that

$$A/(I+J) \simeq (A/I)/\widetilde{J}\,,$$

where $\widetilde{J} = \varphi(J)$ is an ideal in A/I.
(3) Let $\alpha \in \mathbb{C}$ be a root of $X^3 + 2X + 1$ and let $L = \mathbb{Q}(\alpha)$. Recall that $\mathcal{O}_L = \mathbb{Z}[\alpha] \subset L$ (see Exercise 4, Sheet 3).
Using Exercise 1 and question (2) show that

a) (3) is a prime ideal in $\mathcal{O}_L$.
b) (2) is not a prime ideal in $\mathcal{O}_L$.
c) $\rho_1 = (2, \alpha+1)$, $\rho_1 = (2, \alpha^2+\alpha+1)$ are prime ideals in $\mathcal{O}_L$ and $(2) = \rho_1\rho_2$.

Exercise 3. Let F be a field.
(1) Show that $X^3 - Y^2$ is irreducible in $F[X, Y]$.
(2) Denote by A the ring $F[X, Y]/(X^3 - Y^2)$. Show that there is a unique ring homomorphism: $\varphi : A \to F[t]$, such that $\varphi(X) = t^2$, $\varphi(Y) = t^3$. Show that φ is injective. Describe $\varphi(A)$ and show that φ is not surjective.
(3) Show that the field of fractions of $\varphi(A)$ is $F(t)$. Find the integral closure of $\varphi(A)$ in $F(t)$.
(4) Translate the result of (3) in terms of the ring A.

Exercise 4. Recall that for any prime number p the ring $\mathbb{Z}_{(p)}$ is a dvr and $\mathbb{Q}$ is the field of fractions of $\mathbb{Z}_{(p)}$. Let $K = \mathbb{Q}(i)$ be a quadratic field, where $i^2 = -1$. Find all primes p for which the integral closure of $\mathbb{Z}_{(p)}$ in K is a dvr.